

ANALYTIC DEVIATION ONE IDEALS AND TEST MODULES

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ABSTRACT. Let (A, \mathfrak{m}) be a Cohen-Macaulay local ring of dimension $d \geq 1$ and I an ideal in A . Let M be a finitely generated maximal Cohen-Macaulay A -module. Let I be a locally complete intersection ideal with $\text{ht}_M(I) = d - 1$, $l_M(I) = d$ and reduction number at most one. We prove that the polynomial $n \mapsto \ell(\text{Tor}_1^A(M, A/I^{n+1}))$ either has degree $d - 1$ or $F_I(M)$ is a free $F(I)$ -module.

1. INTRODUCTION

Let (A, \mathfrak{m}) be a Cohen-Macaulay local ring of dimension $d \geq 1$ and I an ideal in A . Let M be a finitely generated A -module. Let $F(I) = \bigoplus_{n \geq 0} I^n / \mathfrak{m} I^n$ be the fiber cone of I and $F_I(M) = \bigoplus_{n \geq 0} I^n M / \mathfrak{m} I^n M$ be the fiber module of M with respect to I . Let $l(I) = \dim F(I)$ denote the analytic spread of I . Suppose that $\ell(\text{Tor}_1^A(M, A/I^{n+1})) < \infty$ for all $n \geq 1$. In [6] Kodiyalam proved that there exists a polynomial $t_I^A(M, z) \in \mathbb{Q}[z]$ of degree $\leq l(I) - 1$ such that

$$t_I^A(M, n) = \ell(\text{Tor}_1^A(M, A/I^{n+1})) \quad \text{for } n \gg 0.$$

It is of some interest to find the degree of $t_I^A(M, z)$. In [8, 18] it was proved that if M is a maximal Cohen-Macaulay A -module and $I = \mathfrak{m}$ then

$$\deg t_{\mathfrak{m}}^A(M, z) < d - 1 \quad \text{if and only if } M \text{ is free.}$$

In [5, Theorem I] this result was generalized to arbitrary finitely generated modules with projective dimension at least 1. On the other hand it is easily seen that if $I = (x_1, \dots, x_d)$ is a parameter ideal in A and M is a maximal Cohen-Macaulay A -module then $\text{Tor}_1^A(M, A/I^{n+1}) = 0$ for all $n \geq 0$, see [8, 20].

Assume now that M is a non-free maximal Cohen-Macaulay A -module. Suppose that I is not \mathfrak{m} -primary. Two natural conditions when $\ell(\text{Tor}_1^A(M, A/I^{n+1})) < \infty$ for all $n \geq 0$ are as follows:

- (1) $A_{\mathfrak{p}}$ is a regular local ring for all primes $\mathfrak{p} \neq \mathfrak{m}$.
- (2) $\text{ht}(I) = d - 1$ and I is locally a complete intersection.

We focus our attention on the second condition. We assume that $\text{ht}(I) = d - 1$ and $l(I) = d$ with I a locally complete intersection ideal.

We prove the following theorem

Date: September 5, 2011.

Theorem 1.1. *Let (A, \mathfrak{m}) be a Cohen-Macaulay local ring of dimension $d \geq 1$. Let M be a maximal Cohen-Macaulay A -module. Assume that I is locally a complete intersection ideal in A with $r(I) \leq 1$, $\text{ht}(I) = \text{ht}_M(I) = d - 1$ and $l(I) = d$. If $\deg t_I^A(M, z) < d - 1$ then $F_I(M)$ is a free $F(I)$ -module.*

When (A, \mathfrak{m}) is a hypersurface ring of dimension $d = 1$ we show that $\deg t_I^A(M, z) < d - 1$ if and only if M is free A -module. We also give an example of a non-free maximal Cohen-Macaulay A -module such that $F_I(M)$ is free $F(I)$ -module.

2. PRELIMINARIES

Let (A, \mathfrak{m}) be a local ring with infinite residue field $k = A/\mathfrak{m}$. Let I be an ideal in A and M be a finitely generated A -module.

Lemma 2.1. *Let (A, \mathfrak{m}) be a Cohen-Macaulay local ring of dimension $d \geq 1$. Let M be maximal Cohen-Macaulay A -module. Assume that $I = (x_1, \dots, x_m)$ is an ideal in A generated by an A -regular sequence. Then $\text{Tor}_1^A(M, A/I^n) = 0$ for all $n \geq 1$.*

Proof. The proof is by induction on n . Let $n = 1$. Let L be the first syzygy of M . We have the exact sequence,

$$0 \rightarrow L \rightarrow A^s \rightarrow M \rightarrow 0$$

Since A is Cohen-Macaulay and M is maximal Cohen-Macaulay x_1, \dots, x_m is also an M -regular sequence. Now by [1, 1.1.4] tensoring with A/I gives the following exact sequence,

$$0 \rightarrow L/IL \rightarrow (A/I)^s \rightarrow M/IM \rightarrow 0$$

Thus $\text{Tor}_1^A(M, A/I) = 0$. Now assuming that $\text{Tor}_1^A(M, A/I^n) = 0$ for $n = k$ we prove for $n = k + 1$. Consider the following exact sequence

$$0 \rightarrow I^k/I^{k+1} \rightarrow A/I^{k+1} \rightarrow A/I^k \rightarrow 0$$

Now since $I = (x_1, \dots, x_m)$ is generated by A -regular sequence by [1, 1.1.8] $I^k/I^{k+1} \cong (A/I)^p$ for some $p \geq 1$. Now applying the functor $- \otimes A/I^k$ to the above exact sequence we get the following long exact sequence,

$$\rightarrow \text{Tor}_1^A(M, I^k/I^{k+1}) \rightarrow \text{Tor}_1^A(M, A/I^{k+1}) \rightarrow \text{Tor}_1^A(M, A/I^k) \rightarrow$$

Now $\text{Tor}_1^A(M, I^k/I^{k+1}) \cong \text{Tor}_1^A(M, (A/I)^p) \cong \text{Tor}_1^A(M, A/I)^p = 0$. By induction hypothesis $\text{Tor}_1^A(M, A/I^k) = 0$. Hence $\text{Tor}_1^A(M, A/I^{k+1}) = 0$. \square

We first give two natural conditions when $\ell(\text{Tor}_1^A(M, A/I^{n+1})) < \infty$ for all $n \geq 0$.

Lemma 2.2. *Let (A, \mathfrak{m}) be a Cohen-Macaulay local ring of dimension $d \geq 1$. Let M be non-free maximal Cohen-Macaulay A -module and I be ideal in A . Assume I is not \mathfrak{m} -primary. Then $\ell(\text{Tor}_1^A(M, A/I^{n+1})) < \infty$ for all $n \gg 0$ if $A_{\mathfrak{p}}$ is a regular local ring for all primes $\mathfrak{p} \neq \mathfrak{m}$.*

Proof. Suppose that $A_{\mathfrak{p}}$ is a regular local ring for all primes $\mathfrak{p} \neq \mathfrak{m}$. We first note that if $P \in \text{Supp}(M)$ then M_P is maximal Cohen-Macaulay. Thus when $\mathfrak{p} \neq \mathfrak{m}$ it follows that either $M_P = 0$ or M_P is free. It is now easy to see that $\text{Supp}(\text{Tor}_1^A(M, A/I^{n+1})) \subseteq \{\mathfrak{m}\}$. So $\ell(\text{Tor}_1^A(M, A/I^{n+1})) < \infty$ for all $n \geq 0$. \square

Proposition 2.3. *Let (A, \mathfrak{m}) be a Cohen-Macaulay local ring of dimension $d \geq 1$. Let M be non-free maximal Cohen-Macaulay A -module and I be ideal in A . If $\text{ht}(I) = d-1$, $l(I) = d$ and I is locally a complete intersection then $\ell(\text{Tor}_1^A(M, A/I^{n+1})) < \infty$ for all $n \gg 0$.*

Proof. It suffices to show that $\text{Supp}(\text{Tor}_1^A(M, A/I^{n+1})) \subseteq \{\mathfrak{m}\}$. So let $P \in \text{Supp}(M/IM)$ and consider $(\text{Tor}_1^A(M, A/I^{n+1}))_P = \text{Tor}_1^{A_P}(M_P, A_P/I_P^{n+1})$. If $P \neq \mathfrak{m}$ then by hypothesis I_P is a complete intersection. If $\dim M \geq 2$ then by [8, 20] we have $\text{Tor}_1^{A_P}(M_P, A_P/I_P^{n+1}) = 0$. If $\dim M = 1 = \dim A$ then $\dim M_P = \dim A_P = 0$. So when $P \neq \mathfrak{m}$ we have $I_P^n = 0$ for $n \gg 0$. Hence

$$\text{Tor}_1^{A_P}(M_P, A_P/I_P^{n+1}) = \text{Tor}_1^{A_P}(M_P, A_P) = 0$$

So $\text{Supp}(\text{Tor}_1^A(M, A/I^{n+1})) \subseteq \{\mathfrak{m}\}$. Thus $\ell(\text{Tor}_1^A(M, A/I^{n+1})) < \infty$ for all $n \gg 0$. \square

Remark 2.4. If $l(I) = d-1$ and I is locally a complete intersection then I is a complete intersection.

We need the following lemma.

Lemma 2.5. *Let $R = \bigoplus_{n \geq 0} R_n$ be a Cohen-Macaulay standard graded algebra over an infinite field $k = R_0$. Let $M = \bigoplus_{n \geq 0} M_n$ be a finite graded R -module. Let $x \in R_1$ be M -filter regular. Set $N = M/xM$ and $S = R/(x)$. Assume $\text{depth}(R) \geq 1$. Then*

- (1) *We can choose x such that x is R -regular.*
- (2) *If $\text{grade}(R_+, N) > 0$ then x is M -regular.*
- (3) *If N is free S -module and x is M -regular then M is a free R -module.*

Proof. (i) We first observe that since $\text{depth}(R) \geq 1$ we have $R_+ \notin \text{Ass}(R)$. Let $B = \text{Ass}(R) \cup \text{Ass}(M) \setminus V(R_+)$. Note that R_1 is a vector space over an infinite field k . The set $C = \{P \cap R_1 \mid P \in B\}$ is a finite set consisting of proper subspaces of R_1 . So we can choose an element $x \in R_1 \setminus \bigcup_{P \in C} P \cap R_1$. It is now easy to see that x is R -regular and M -filter regular.

(ii) Let $N = M/xM$ and $B = (0 :_M x)$. Since $B_n = 0$ for $n \gg 0$, $H_{R_+}^i(B) = 0$ for $i > 0$. So we have the following long exact sequence

$$0 \longrightarrow H_{R_+}^0(B) \longrightarrow H_{R_+}^0(M)(-1) \xrightarrow{\mu_x} H_{R_+}^0(M) \longrightarrow H_{R_+}^0(N)$$

$$\longrightarrow H_{R+}^1(M)(-1) \xrightarrow{\mu_x} H_{R+}^1(M) \longrightarrow H_{R+}^1 R_+(N) \\ \dots$$

Since $\text{grade}(R_+, N) > 0$ we have $H_{R+}^0(N) = 0$. Hence the exact sequence above becomes

$$0 \longrightarrow H_{R+}^0(B) \longrightarrow H_{R+}^0(M)(-1) \xrightarrow{\mu_x} H_{R+}^0(M) \longrightarrow 0$$

Now note that $H_{R+}^0(M)$ is a finite dimensional vector space. Thus μ_x surjective gives $H_{R+}^0(M) \cong H_{R+}^0(M)(-1)$. So $H_{R+}^0(M) = 0$. Therefore $\text{grade}(R_+, M) > 0$. It follows that x is M -regular.

(iii) Let $l = \mu(M)$. Note that $l = \mu(M/xM)$. Now consider the following exact sequence:

$$0 \longrightarrow L \longrightarrow R^l \longrightarrow M \longrightarrow 0$$

where L is the first syzygy of M . Now since x is M -regular it follows from [1, 1.1.4] that

$$0 \longrightarrow L/xL \longrightarrow S^l \longrightarrow N \longrightarrow 0$$

is exact sequence. By hypothesis N is free S -module and so $N \cong S^l$. Thus $L/xL = 0$. By graded Nakayama lemma we get that $L = 0$. Thus $M \cong R^l$ and hence a free R -module. \square

We recall the definition of superficial element.

Definition 2.6. An element $x \in I$ is I -superficial for M if there exists a positive integer c with

$$(I^{n+1}M :_M x) \cap I^c M = I^n M \text{ for all } n \geq c.$$

Remark 2.7. Assume that $\text{grade}(I, M) \geq 1$. If an element is I -superficial on M then x is regular on M and

$$(I^{n+1}M :_M x) = I^n M \text{ for all } n \gg 0.$$

A convention: The degree of the zero polynomial is defined to be $-\infty$.

Lemma 2.8. Let (A, \mathfrak{m}) be a local ring and M a finite non-free A -module. Let I be an ideal in A . Denote by L the first syzygy of M . Let $x \in I$ be a superficial non-zero divisor on A , M , and L . Suppose that $\ell(\text{Tor}_1^A(M, A/I^{n+1})) < \infty$ for $n \gg 0$. Set $B = A/xA$, $N = M/xM$ and $J = I/(x)$. Then we have

$$t_I^A(M; n) = t_I^A(M; n-1) + t_J^B(N; n) \text{ for all } n \gg 0.$$

$$\deg t_J^B(N; z) \leq \deg t_I^A(M; z) - 1.$$

Proof. Since x is I -superficial for A , one has following exact sequence for all $n \gg 0$

$$0 \longrightarrow A/I^n \xrightarrow{i} A/I^{n+1} \longrightarrow B/J^{n+1} \longrightarrow 0$$

where the map i is defined by $i_n(a + I^n) = xa + I^{n+1}$. Applying $M \otimes_A -$ to above exact sequence gives the following exact sequence of A -modules

$$\begin{aligned} \mathrm{Tor}_1^A(M, A/I^n) &\xrightarrow{\mathrm{Tor}_1^A(M, i_n)} \mathrm{Tor}_1^A(M, A/I^{n+1}) \longrightarrow \mathrm{Tor}_1^A(M, B/J^{n+1}) \longrightarrow \\ M/I^n M &\xrightarrow{M \otimes i_n} M/I^{n+1} M \longrightarrow N/J^{n+1} N \longrightarrow 0. \end{aligned}$$

Since x is I -superficial on M the map $M \otimes i_n$ is injective for $n \gg 0$. We claim that the map $\mathrm{Tor}_1^A(M, i_n)$ is injective for $n \gg 0$. For this consider the exact sequence defining L , i.e.

$$0 \longrightarrow L \longrightarrow F \longrightarrow M \longrightarrow 0.$$

Now applying the functors $-\otimes_A A/I^n$ and $-\otimes_A A/I^{n+1}$ one gets the following commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathrm{Tor}_1^A(M, A/I^n) & \longrightarrow & L/I^n L \\ & & \downarrow \mathrm{Tor}_1^A(M, i_n) & & \downarrow L \otimes i_n \\ 0 & \longrightarrow & \mathrm{Tor}_1^A(M, A/I^{n+1}) & \longrightarrow & L/I^{n+1} L \end{array}$$

Notice the following

$$\mathrm{Ker}(L \otimes i_n) = \frac{I^{n+1}L : x}{I^n L}.$$

As x is superficial on L it follows that the map $L \otimes i_n$ is injective for $n \gg 0$. Thus $\mathrm{Tor}_1^A(M, i_n)$ is injective for $n \gg 0$.

So for $n \gg 0$ above long exact sequence becomes

$$0 \longrightarrow \mathrm{Tor}_1^A(M, A/I^n) \longrightarrow \mathrm{Tor}_1^A(M, A/I^{n+1}) \longrightarrow \mathrm{Tor}_1^A(M, B/J^{n+1}) \longrightarrow 0.$$

Now since x is both R -regular and M -regular we obtain the following isomorphism, see [7, 18.2]

$$\mathrm{Tor}_1^A(M, B/J^{n+1}) \cong \mathrm{Tor}_1^B(N, B/J^{n+1}).$$

From this isomorphism and the exact sequence above it follows that for $n \gg 0$

$$\ell_A(\mathrm{Tor}_1^A(M, A/I^{n+1})) = \ell_A(\mathrm{Tor}_1^A(M, A/I^n)) + \ell_A(\mathrm{Tor}_1^B(N, B/J^{n+1})).$$

Thus it follows that

$$\deg t_J^B(N; z) \leq \deg t_I^A(M; z) - 1.$$

□

Recall that an ideal $J \subseteq I$ is reduction of I if there exists a natural number m such that $J I^n = I^{n+1}$ for all $n \geq m$. We define $r_J(I)$ to be the least such m . A reduction J of I is called minimal if it is minimal with respect to inclusion. Reduction number of I is defined as follows,

$$r(I) = \min\{r_J(I) \mid J \text{ is minimal reduction of } I\}.$$

Proposition 2.9. *Let (A, \mathfrak{m}) be a Cohen-Macaulay local ring of dimension $d \geq 2$ with infinite residue field $k = A/\mathfrak{m}$. Let M be a maximal Cohen-Macaulay A -module. Assume I is locally a complete intersection ideal in A with $\text{ht}(I) = \text{ht}_M(I) = d - 1$, $r(I) \leq 1$ and $l_M(I) = d$. Denote by L the first syzygy of M . Then we can choose $x \in I$ satisfying the following conditions*

- (i) x^* is $G_I(A)$ -regular.
- (ii) x^* is $G_I(M)$ -regular.
- (iii) x^* is $G_I(L)$ -filter regular.
- (iv) x° is $F(I)$ -regular.
- (v) x° is $F_I(M)$ -filter regular.

Proof. Since $r(I) \leq 1$ by [2, 4.2] $F(I)$ is Cohen-Macaulay. Note here that $l_M(I) = d$ gives $l(I) = d$. Now by [3, 2.9] $G_I(A)$ is Cohen-Macaulay ring. Extension of Huckaba-Huneke's result to maximal Cohen-Macaulay modules gives $G_I(M)$ is a Cohen-Macaulay $G_I(A)$ module. Let $\mathfrak{A} = \text{Ass}(G_I(A)) \cup \text{Ass}(G_I(M))$. Note that since $d \geq 2$ we have $\text{grade}(I) = d - 1 \geq 1$. So if $P \in \text{Ass}(G_I(A)) \cup \text{Ass}(G_I(M))$ then $P \notin V(G(I)_+)$. Hence

$$\mathfrak{A} = \text{Ass}(G_I(A)) \cup \text{Ass}(G_I(M)) \setminus V(G(I)_+).$$

Consider the following set

$$\mathfrak{B} = \mathfrak{A} \cup \text{Ass}(G_I(L)) \setminus V(G(I)_+).$$

Let $\mathfrak{B} = \{P_1, \dots, P_t\}$. Set $V_i = P_i \cap I/I^2$. It is clear that V_i are proper submodules of I/I^2 . Since $F(I)$ is Cohen-Macaulay we have

$$\mathfrak{C} = \text{Ass}(F(I)) = \text{Ass}(F(I)) \setminus V(F(I)_+).$$

Now let $\mathfrak{D} = \mathfrak{C} \cup \text{Ass}(F_I(M)) \setminus V(F(I)_+) = \{Q_1, \dots, Q_k\}$. Set $W_i = Q_i \cap I/\mathfrak{m}I$. Observe that $V_i \otimes k$ and W_j are proper subspaces of $I/\mathfrak{m}I$ for $1 \leq i \leq t$ and $1 \leq j \leq k$. Since k is infinite we can choose

$$\bar{x} \in \frac{I}{\mathfrak{m}I} \setminus \left\{ \bigcup_{i=1}^t V_i \otimes k, \bigcup_{i=1}^k W_i \right\}.$$

We now verify conditions (i) to (v). Since $x \notin V_i \otimes k$ gives that $x \notin V_i$. Therefore $x^* \notin P$ for $P \in \mathfrak{A}$. Hence x^* is $G_I(A)$ -regular and $G_I(M)$ -regular. So (i) and (ii) follow. Similarly (iii) follows because $x^* \notin P$ for $P \in \mathfrak{B}$. For (iv) and (v) we notice that $x^\circ \notin P$ for $P \in \mathfrak{D}$ since $x \notin W_i$. It therefore follows that x° is $F(I)$ -regular and $F_I(M)$ -filter regular. \square

Corollary 2.10. *Let (A, \mathfrak{m}) be a Cohen-Macaulay local ring of dimension $d \geq 2$ with infinite residue field $k = A/\mathfrak{m}$. With I , L , M and $x \in I$ as in the Proposition 2.9 we have*

- (i) x is I -superficial on A , M and L .

(ii) x is regular on A , M and L .

Proof. (1) This follows easily from the first three conditions of the Proposition 2.9.
 (2) Note that grade of I w.r.t. R , M and L are all at least 1. So superficial elements on them are regular elements. \square

Remark 2.11. For $x \in I$ as chosen in Proposition 2.9 we have $\bar{I} = I/(x)$ is locally a complete intersection ideal in $\bar{A} = A/(x)$. To see this it suffices to observe that $\text{Min}(\bar{A}/\bar{I}) = \{P/(x) \mid P \in \text{Min}(A/I)\}$.

Remark 2.12. If J is minimal reduction of I such that J is generated by a regular sequence and $JI = I^2$ then $F(I)$ is Cohen-Macaulay, see [9, 1]. This result holds true even when I is locally a complete intersection, see [2, 4.2].

Lemma 2.13. Let $x \in I$ and set $B = A/(x)$, $N = M/xM$ and $\bar{I} = I/(x)$. Suppose x^* is $G_I(M)$ -regular then we have the following isomorphism

$$F_{\bar{I}}(N) \cong \frac{F_I(M)}{x^o F_I(M)}.$$

Proof. Observe that we have

$$F_{\bar{I}}(N)_n = \frac{I^n M + (x)M}{\mathfrak{m}I^n M + (x)M} \quad \text{and} \quad \left[\frac{F_I(M)}{x^o F_I(M)} \right]_n = \frac{I^n M}{xI^{n-1}M + \mathfrak{m}I^n M}$$

Consider the following natural map

$$\alpha : I^n M \longrightarrow \frac{I^n M + (x)M}{\mathfrak{m}I^n M + (x)M}$$

We now have

$$\text{Ker} \alpha = I^n M \cap (\mathfrak{m}I^n M + (x)M) = \mathfrak{m}I^n M + (x)M \cap I^n M$$

Since x^* is $G_I(M)$ -regular for all $n \geq 1$ we have $(x)M \cap I^n M = xI^{n-1}M$ by Valabrega-Valla criterion, see [10, 2.6]. Hence $\text{Ker} \alpha = xI^{n-1}M + \mathfrak{m}I^n M$ and so

$$\frac{I^n M + (x)M}{\mathfrak{m}I^n M + (x)M} \cong \frac{I^n M}{xI^{n-1}M + \mathfrak{m}I^n M}$$

Hence

$$F_{\bar{I}}(N) \cong \frac{F_I(M)}{x^o F_I(M)}$$

\square

3. PROOF OF THE MAIN THEOREM

We now prove the main theorem.

Theorem 3.1. Let (A, \mathfrak{m}) be a Cohen-Macaulay local ring of dimension $d \geq 1$. Let M be a maximal Cohen-Macaulay A -module. Assume I is locally a complete intersection ideal in A with $r(I) \leq 1$, $\text{ht}(I) = \text{ht}_M(I) = d - 1$ and $l_M(I) = d$. If $\deg t_I^A(M, z) < d - 1$ then $F_I(M)$ is a free $F(I)$ -module.

Proof. It follows from Proposition 2.3 that $\ell(\text{Tor}_1^A(M, A/I^{n+1})) < \infty$ for all $n \geq 0$. So $\deg t_I^A(M, z) \leq d-1$. Suppose now that $\deg t_I^A(M, z) < d-1$. Note that $l_M(I) = d$ gives $l(I) = d$. We prove the result by induction on d .

Suppose first $d = 1$. Let $J = (a)$ be a minimal reduction of I . Since $r(I) \leq 1$ by [2, 4.2] $F(I)$ is Cohen-Macaulay. Now since $r(I) \leq 1$ and $\dim F(I) = 1$ it follows from [1, 4.1.12] that $\mu(I^n) = c$ for all $n \geq 1$. Since $\deg t_I^A(M, z) < 0$ we have $\text{Tor}_1^A(M, A/I^n) = 0$ for $n \gg 0$. So $M \otimes_A I^n = I^n M$. Tensoring with A/\mathfrak{m} gives the following

$$\frac{M}{\mathfrak{m}M} \otimes_k \frac{I^n}{\mathfrak{m}I^n} = \frac{I^n M}{\mathfrak{m}I^n M}.$$

Therefore

$$(1) \quad \ell\left(\frac{I^n M}{\mathfrak{m}I^n M}\right) = c\mu(M) \quad \text{for all } n \gg 0.$$

Claim: $F_I(M)$ is a free $F(I)$ -module.

Note that we have $aI^n M = I^{n+1}M$ for $n \geq 1$. Therefore

$$c\mu(M) \geq \mu(IM) \geq \mu(I^2 M) \geq \dots \geq \mu(I^n M) \geq \dots$$

But we have $\mu(I^n M) = c\mu(M)$ so that $\mu(I^n M) = c\mu(M)$ for all $n \geq 1$.

Let $\mu(M) = t$ and $M = \langle m_1, \dots, m_t \rangle$. We define a graded $F(I)$ -linear map ϕ

$$\phi : F(I)^t \longrightarrow F_I(M)$$

by $\phi(e_i) = \bar{m}_i$ where $\bar{m}_i \in M/\mathfrak{m}M$. Observe that ϕ is surjective,

$$\dim_k [F_I(M)^t]_0 = t$$

$$\dim_k [F_I(M)]_0 = t$$

and for $n \geq 1$

$$\dim_k [F(I)^t]_n = ct$$

$$\dim_k [F_I(M)]_n = ct$$

Therefore ϕ is an isomorphism. So $F_I(M)$ is free $F(I)$ -module.

Suppose now that $d \geq 2$. So $\text{ht}(I) = d-1 \geq 1$. Choose $x \in I$ as in Proposition 2.9. So now x satisfies all the conditions given in Proposition 2.9 and Corollary 2.10. Set $B = A/(x)$, $N = M/xM$ and $\bar{I} = I/(x)$. By our choice of x we have $\text{ht}(\bar{I}) = \text{ht}_N(\bar{I}) = d-2$ and $l_N(\bar{I}) = d-1$. By Remark 2.11 \bar{I} is locally a complete intersection in B . Also $r(\bar{I}) \leq 1$. Now by Lemma 2.8 we have

$$\deg t_{\bar{I}}^B(N; z) \leq \deg t_I^A(M; z) - 1.$$

By our hypothesis $\deg t_I^A(M; z) < d-1$ and so we have $\deg t_{\bar{I}}^B(N; z) < d-2$. Thus by induction hypothesis we have $F_{\bar{I}}(N)$ is free $F(\bar{I})$ -module. Note that we have following isomorphisms by Lemma 2.13

$$F_{\bar{I}}(N) \cong \frac{F_I(M)}{x^\circ F_I(M)} \quad \text{and} \quad F(\bar{I}) \cong \frac{F(I)}{x^\circ F(I)}$$

Since $F_{\bar{I}}(N)$ is free $F(\bar{I})$ -module one has $\text{depth}(F_{\bar{I}}(N)) = \text{depth}(F(\bar{I}))$. Now since $F(I)$ is Cohen-Macaulay and x^o is $F(I)$ -regular we have $\text{depth}(F(\bar{I})) = d - 1 \geq 1$. Hence $\text{depth}(F_{\bar{I}}(N)) \geq 1$. By our choice of x we have that x^o is $F_I(M)$ -filter regular. So x^o is $F_I(M)$ -regular by Lemma 2.5(2). Again by Lemma 2.5(3) we get that $F_I(M)$ is free $F(I)$ -module. \square

4. THE CASE OF HYPERSURFACE

When A is a hypersurface ring we show that if M is non-free maximal Cohen-Macaulay A -module of dimension $d = 1$ having constant rank then $\deg t_I(M, z) = d - 1$.

Proposition 4.1. *Let A be a hypersurface ring of dimension $d = 1$. Let M be a maximal Cohen-Macaulay A -module having constant rank. Let I be a locally complete intersection ideal with $\text{ht}(I) = 0$ and $l(I) = 1$. Then the following conditions are equivalent:*

- (i) $\deg t_I^A(M, z) < 0$.
- (ii) M is free A -module.

Proof. So let $d = 1$ first. Consider the following exact sequence

$$0 \longrightarrow I^n \longrightarrow A \longrightarrow A/I^n \longrightarrow 0$$

Tensoring with M we get

$$0 \longrightarrow \text{Tor}_1^A(M, A/I^n) \longrightarrow M \otimes_A I^n \longrightarrow M \longrightarrow M/I^n M \longrightarrow 0$$

Suppose $\deg t_I(M, z) < d - 1 = 0$. So $\text{Tor}_1^A(M, A/I^n) = 0$. Since $\dim M = 1$ we obtain from the exact sequence above that $M \otimes_A I^n$ is maximal Cohen-Macaulay A -module. It now follows from the [4, Theorem 3.1] that atleast one of M or I^n is free A -module. But I^n cannot be free because $\text{ht}(I) = 0$. Hence M is free A -module.

(ii) \Rightarrow (i) is obvious. \square

5. EXAMPLE

We give an example where $F_I(M)$ is one dimensional non-free $F(I)$ -module having depth zero.

Example 5.1. Let A be a Cohen-Macaulay local ring of dimension 1 with atleast two distinct minimal prime ideals. Let P_1 and P_2 be those two minimal prime ideals. Now let $M = A/P_1 \oplus A/P_2$. Notice M is maximal Cohen-Macaulay A -module of dimension 1. Now choose $b \in P_2 \setminus P_1$. Let $J = (b)$ and $\mathfrak{A} = \text{Min}(A/J) = \{P_2, P_3, \dots, P_s\}$. Since A_{P_i} is Artinian we have $s_i b^{n_i} = 0$ for $s_i \notin P_i$ and for $1 \leq i \leq s$. Let $n = \max\{n_i \mid 1 \leq i \leq s\}$. Let $a = b^n$ and $I = (a)$. We now have $I_P = 0$ for $P \in \text{Min}(A/I)$. We thus have

- (i) $\text{ht}(I) = 0$.
- (ii) $l(I) = 1$.
- (iii) I is locally a complete intersection.
- (iv) $\mu(M) = 2$.
- (v) $\mu(a^n M) = \ell(a^n M / \mathfrak{m} a^n M) = 1$ for all $n \geq 1$ (Since $a^n \notin P_1$ for all $n \geq 1$).
- (vii) M is not free.

Computation of Tor:

$$\text{Tor}_1^A(M, A/I^n) = \text{Tor}_1^A(A/P_1 \oplus A/P_2, A/(a^n)) \cong \frac{P_2 \cap (a^n)}{P_2(a^n)}$$

Since $a \in P_2$ we have $P_2 \cap (a^n) = (a^n)$. So

$$\text{Tor}_1^A(M, A/I^n) \cong \frac{(a^n)}{P_2(a^n)}$$

By Nakayama Lemma $(a^n) \neq P_2(a^n)$ so that for $n \gg 0$ we have

$$\ell(\text{Tor}_1^A(M, A/I^n)) = c$$

where $c > 0$. Notice now that $F(I) \cong k[x]$. We claim that $F_I(M)$ is not free. Supposing otherwise we get that $F_I(M) \cong k[x]^t$ for some $t \geq 1$. Since all the generators of $F_I(M)$ are in degree zero we get that $\mu(M) = 1$ which is contradiction to (iv) above. So $F_I(M)$ is not free $F(I)$ -module. Since $k[x]$ is regular ring we have $\text{Proj } F_I(M) < \infty$. By Auslander-Buchsbaum formula we have $\text{depth}(F_I(M)) = 0$. So $F_I(M)$ is not Cohen-Macaulay $F(I)$ -module.

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